On the Direct Cauchy Theorem in Widom Domains: Positive and Negative Examples

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Abstract

We discuss several questions which remained open in our joint work with M. Sodin "Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character–automorphic functions". In particular, we show that there exists a non-homogeneous set E such that the Direct Cauchy Theorem (DCT) holds in the Widom domain $\mathbb{C} \setminus E$. On the other hand we demonstrate that the weak homogeneity condition on E (introduced recently by Poltoratski and Remling) does not ensure that DCT holds in the corresponding Widom domain.

1 Introduction

Several recent publications [3, 7, 9, 10, 15, 18] indicate a certain interest in our paper [19]. We recall the main result of this paper, simultaneously we introduce notations and give definitions necessary in what follows.

Let E be a compact on the real axis without isolated points, $E = [b_0, a_0] \setminus \cup_{j \geq 1}(a_j, b_j)$. By $J(E)$ we denote the set of reflectionless (two-sided) Jacobi matrices J with the spectrum on E. This means that the diagonal elements of the resolvent

$$R_{k,k}(z) = \langle (J - z)^{-1} e_k, e_k \rangle$$

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possess the property $\operatorname{Re} R_{k,k}(x+i0) = 0$ for almost all $x \in E$. As usual e_k's denote the standard basis in l^2. For an exceptional role of this class of Jacobi matrices see [17].

The function $R(z) = R_{k,k}(z)$ has positive imaginary part in the upper half plane, and therefore possesses the representation

$$R(z) = \int_E \frac{d\sigma(x)}{x-z}. \tag{1.1}$$

Moreover, since $R(z)$ assumes *pure imaginary values* on E we have

$$\sigma'_{a.c.} = \frac{1}{\pi}|R(x)|.$$

We follow the terminology in [15] and call reflectionless the measures σ related to reflectionless functions $R(z)$ (1.1), $\operatorname{Re} R(x+i0) = 0$, a.e. $x \in E$.

The collection of reflectionless functions associated to the given compact E can be parameterized in the following way. We chose arbitrary $x_j \in [a_j, b_j]$ and set

$$R(z) = R(z, \{x_j\}) = -\frac{1}{\sqrt{(z-a_0)(z-b_0)}} \prod_{j\geq 1} \frac{z-x_j}{\sqrt{(z-a_j)(z-b_j)}}. \tag{1.2}$$

By $D(E)$ we denote the set of so-called divisors D, where

$$D = \{(x_j, \epsilon_j) : \ x_j \in [a_j, b_j],\ \epsilon_j = \pm 1\}, \quad (a_j, 1) \equiv (a_j, -1),\ (b_j, 1) \equiv (b_j, -1). \tag{1.3}$$

The map $J(E) \to D(E)$ is defined in the following way. For a reflectionless J the resolvent function $R_{0,0}(z)$ possesses the representation (1.2) and this representation produces the collection $\{x_j\}$. To define ϵ_j we represent J as a two-dimensional perturbation of the block-diagonal sum of one-sided Jacobi matrices $J_\pm$, that is,

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + p_0 e_{-1}\langle \cdot, e_0\rangle + p_0 e_0 \langle \cdot, e_{-1}\rangle.$$

This representation generates the identity

$$-\frac{1}{R_{0,0}(z)} = -\frac{p_0^2}{r_-(z)} + r_+(z), \tag{1.4}$$

where

$$r_+(z) = \langle (J_+ - z)^{-1} e_0, e_0\rangle, \quad r_-(z) = \langle (J_- - z)^{-1} e_{-1}, e_{-1}\rangle.$$

For $x_j \in (a_j, b_j)$ this point is a pole of only one of the two functions in the right hand side of (1.4). We set $\epsilon_j = 1$ if x_j is a pole of $r_+(z)$ and $\epsilon_j = -1$ in the opposite case.

Next we will define the so called generalized Abel map, the map from the collection of divisors $D(E)$ to the group of characters of the fundamental group of the domain $\Omega = \bar{\mathbb{C}} \setminus E$. We are able to do this for Widom domains.

Let $z : \mathbb{D}/\Gamma \to \Omega$ be a uniformization of the domain Ω, that is, an analytic function F in Ω can be represented by an analytic function f in $\mathbb{D}$, which is automorphic with respect to the action of the Fuchsian group Γ:

$$F(z(\zeta)) = f(\zeta), \ f(\gamma(\zeta)) = f(\zeta), \ \zeta \in \mathbb{D}, \ \gamma \in \Gamma.$$

The dual group Γ^* is formed by characters

$$\alpha : \Gamma \to \mathbb{T} \text{ such that } \alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2), \ \gamma_{1,2} \in \Gamma.$$

For a character α the Hardy space $H^p(\alpha)$ is the subspace of the standard H^p consisting of character automorphic functions $f(\gamma(\zeta)) = \alpha(\gamma) f(\zeta)$.

A domain is called of Widom type if $H^\infty(\alpha)$ is not trivial (contains a non constant function) for all $\alpha \in \Gamma^*$. Let $b(\zeta, \zeta_0)$ be the Green function of the group Γ [16], i.e.,

$$b(\zeta, \zeta_0) = \prod_{\gamma \in \Gamma} \frac{\gamma(\zeta_0) - \zeta}{1 - \zeta\overline{\gamma(\zeta_0)}} \frac{|\gamma(\zeta_0)|}{\gamma(\zeta_0)} \tag{1.5}$$

Note that $G(z(\zeta), z_0) := -\log|b(\zeta, \zeta_0)|$ is the Green function in the domain Ω. We assume that $z(0) = \infty$, that is, $G(z) = -\log|b(\zeta, 0)|$ is the Green function with respect to infinity.

A regular domain Ω is of Widom type if and only if

$$\sum_{c_k : \nabla G(c_k) = 0} G(c_k) < \infty. \tag{1.6}$$

Let us point out that the critical point c_k belongs to (a_k, b_k). The condition (1.6) guaranties that the following product

$$K^D(\zeta) = \sqrt{\prod_{j \geq 1} \frac{z(\zeta) - x_j}{z(\zeta) - c_j} \frac{b(\zeta, c_j)}{b(\zeta, x_j)}} \prod_{j \geq 1} b(\zeta, x_j)^{\frac{1+\epsilon_j}{2}} \tag{1.7}$$

converges for an arbitrary $D \in D(E)$.

The Abel map $D(E) \to \Gamma^*$ is defined by the relation

$$K^D(\gamma(\zeta)) = \alpha^D(\gamma) K^D(\zeta).$$

The third map $\Gamma^* \to J(E)$ is defined as follows. Let k^α be the reproducing kernel in $H^2(\alpha)$ with respect to the origin, i.e., $\langle f, k^\alpha \rangle = f(0)$ for all $f \in H^2(\alpha)$. Let α_0 be the character of the Green function b, $b \circ \gamma = \alpha_0(\gamma) b$. Then $J(\alpha)$ is the matrix of the multiplication operator by $z(\zeta)$ with respect to the orthonormal basis $\{e(\zeta, n)\}_{n \in \mathbb{Z}}$ in $L^2(\alpha)$, where

$$e(\zeta, n) = b^n \frac{k^{\alpha\alpha_0^{-n}}}{\sqrt{k^{\alpha\alpha_0^{-n}}(0)}}.$$

The main result in [19] claims that under a certain additional condition on the domain Ω all three maps

$$J(E) \to D(E) \to \Gamma^* \to J(E) \tag{1.8}$$

are one-to-one and continuous with respect to the operator norm topology in $J(E)$; recall that Γ^* is a compact Abelian group; $D(E)$ is equipped with the product topology.

This additional condition is called the *Direct Cauchy Theorem* (DCT) [11]. In fact, it is not a theorem, but a certain property of a Widom domain. For some of them DCT holds true, for others it fails.

We say that $F(z)$ is of Smirnov class in Ω if the corresponding function $f(\zeta) = F(z(\zeta))$ is of Smirnov class in $\mathbb{D}$, that is, it possesses a representation $f = \frac{f_1}{f_2}$, where f_1, f_2 are uniformly bounded, moreover the denominator is an outer function.

Definition 1.1. The space $E_0^1(\Omega)$ is formed by Smirnov class functions F in Ω such that

$$\|F\| = \frac{1}{2\pi} \int_E |F(x + i0)| \, dx + \frac{1}{2\pi} \int_E |F(x - i0)| \, dx < \infty,$$

and $F(\infty) = 0$.

Definition 1.2. We say that the *Direct Cauchy Theorem (DCT)* holds if

$$\frac{1}{2\pi i} \oint_E F(x) dx = \mathrm{Res}_\infty F(z) dz = A, \quad F(z) = -\frac{A}{z} + \dots, \ z \to \infty, \tag{1.9}$$

for all $F \in E_0^1(\Omega)$.

Theorem 1.3 (Sodin-Yuditskii [19]). *Let E be such that $\Omega = \bar{\mathbb{C}} \setminus E$ is of Widom type with DCT. Then every Jacobi matrix $J \in J(E)$ is almost periodic.*

The following notion was introduced by Carleson [8]. A compact E is homogeneous if there exists $\eta = \eta(E) > 0$ such that

$$|E \cap (x - \delta, x + \delta)| \geq \eta\delta \tag{1.10}$$

for all $x \in E$ and $\delta \in (0, 1)$.

It was noted in [19] that if E is homogeneous then Ω is a Widom domain with DCT. Thus the homogeneity is a very nice constructive condition on E that guaranties that $J(E)$ consists of almost periodic operators. For instance all standard Cantor sets of positive length are homogeneous, for a proof see e.g. [14].

In this note we answer several remaining open questions in the above context.

First, the map $J(E) \to D(E)$ is one-to-one if and only if all reciprocal $-1/R(z)$ to reflectionless (Nevanlinna class) functions have no singular component on the set E in their integral representation, i.e., for

$$-\frac{1}{R(z)} = z - q_0 + \sum_{x_j \in (a_j, b_j)} \frac{\tau_j}{x_j - z} + \int_E \frac{d\tau(x)}{x - z} \tag{1.11}$$

the singular component of the measure τ is trivial, $\tau_s(E) = 0$.

The fact that neither reflectionless measure nor its reciprocal (1.11) has no singular component on E for Widom domains with DCT was proved in [14].

In general, the question on the possible support of the singular part of a reflectionless measure was studied in [15]. In particular they introduced a notion of a weakly homogeneous set: a Borel set E is weakly homogeneous if

$$\limsup_{\delta \to +0} \frac{1}{\delta} |E \cap (x - \delta, x + \delta)| > 0, \quad x \in E, \tag{1.12}$$

and proved that all reflectionless measures on a weakly homogeneous set are absolutely continuous.

Moreover, previously known examples of Widom domains such that DCT fails were based on the idea to construct a reflectionless measure with a non-trivial singular component, for details see Sect. 4.

So, assume a priory that $\Omega = \bar{\mathbb{C}} \setminus E$ is of Widom type.

- Does the weak homogeneity (1.12) imply DCT in this case?

Note that this assumption (1.12) is even stronger than the property that all reflectionless measures associated with the given E (and their reciprocals (1.11)) are absolutely continuous.

Second, the map $D(E) \to \Gamma^*$, which was defined for Widom domains, is one-to-one if and only if DCT holds. It deals with the following property of H^2-spaces in Ω.

In the classical Hardy spaces theory there is an important description of their orthogonal complement in the standard L^2, namely, $\bar{f} \in L^2 \ominus H^2$ if and only if $f \in H^2$ and $f(0) = 0$. In the Widom domain case the following statement holds true. Define the Blaschke product

$$\theta(\zeta) = \prod_{j \geq 1} b(z, c_j),$$

which converges due to the Widom condition (1.6), and denote by α_θ the corresponding character, $\theta \circ \gamma = \alpha_\theta(\gamma)\theta$. Then, if $\bar{f} \in L^2(\alpha) \ominus H^2(\alpha)$, then $\theta f \in H^2(\alpha_\theta \alpha^{-1})$ and $f(0) = 0$. In other words, if we define

$$\check{H}^2(\alpha) = \{f \in L^2(\alpha) : \ \theta \bar{f} \in L^2(\alpha_\theta \alpha^{-1}) \ominus H^2_0(\alpha_\theta \alpha^{-1})\},$$

then $\check{H}^2(\alpha) \subset H^2(\alpha)$.

Thus, generally speaking, for a Widom domain to the given character α one can associate the biggest $H^2(\alpha)$ and the smallest $\check{H}^2(\alpha)$ possible Hardy spaces. They coincide, i.e., $H^2(\alpha) = \check{H}^2(\alpha)$ for all $\alpha \in \Gamma^*$, if and only if DCT holds in the Widom domain [19]. Both reproducing kernels $k^\alpha / \|k^\alpha\|$ and $\check{k}^\alpha / \|\check{k}^\alpha\|$ are functions of the form (1.7). Thus, as soon as $H^2(\alpha) \neq \check{H}^2(\alpha)$ two different divisors in $D(E)$ correspond to the same character α.

- How big can the defect subspace $H^2(\alpha) \ominus \check{H}^2(\alpha)$ be?

Finally, it is worthwhile to clarify:

- Is the homogeneity (1.10) just a sufficient condition for DCT or is it also necessary?

To answer these three questions we first relate DCT with an L^1-extremal problem, Sect. 2. In Sect. 3 we reveal the structure of the extremal function, Theorem 3.3. It shows, that in the simplest case, when E consists of a system of intervals having a unique accumulation point $x_0 \in E$, our questions can be reduced to approximation problems for *entire functions* (with respect

to the variable $1/(z - x_0)$), e.g., see Lemma 6.1. This area was developed essentially recently by Borichev-Sodin [4, 5, 6].

We can summarize the results of the last two sections in the following proposition.

Proposition 1.4. *Define the following three classes of Widom(-Denjoy) domains:*

- $\Omega = \bar{\mathbb{C}} \setminus E \in W_{hom}$ *if* E *is homogeneous;*
- $\Omega \in W_{DCT}$ *if Direct Cauchy Theorem holds in* Ω*;*
- $\Omega \in W_{a.c}$ *if all reflectionless measures* (1.1) *in* Ω *and their reciprocal* (1.11) *are absolutely continuous on* E.

Then

$$W_{hom} \subset W_{DCT} \subset W_{a.c.} \tag{1.13}$$

and both inclusions are proper.

Acknowledgment. Initial push to consider these problems was generated by a small discussion with Barry Simon. Then we discussed the subject with Sasha Volberg, the manuscript [13] is the result of this discussion. I have to mention a special role of Misha Sodin: I am very thankful for his explanations dealing with the potential theory in the Denjoy domains, in particular of Benedicks' construction used in section 6.

2 L^1 extremal problem in Widom Domain and Direct Cauchy Theorem

We consider here the following extremal problem.

Problem 2.1. Find

$$M = \inf\{\|F\| : F \in E_0^1(\Omega), \quad F(z) = -\frac{1}{z} + \dots, \ z \to \infty\}. \tag{2.1}$$

This extremal problem is closely related with the DCT.

Theorem 2.2. *For a Widom domain* Ω *the DCT holds if and only if* $M = 1$ *in Problem* 2.1.

Proof. We adapt the general proof of this kind of theorem [11] to the special Denjoy domain case. Let $R(z)$ be a reflectionless function, see (1.2). It was shown in [19] that $R(z)$ is of Smirnov class, so $R \in E^1_0(\Omega)$ and we get

$$M \leq \|R\| \leq \int_E d\sigma = 1. \tag{2.2}$$

On the other hand, if DCT holds, then for every function F of the set (2.1) we have

$$1 = \frac{1}{2\pi i} \oint_E F \leq \|F\|.$$

Thus $M = 1$.

Conversely, $M = 1$ implies

$$|\Lambda(F)| \leq \|F\|$$

for the functional

$$\Lambda(F) = A,\ F \in E^1_0(\Omega),\ F = -\frac{A}{z} + \dots,\ z \to \infty.$$

Since $E^1 \subset L^1$ we can extend Λ to a functional in L^1. Therefore there exists $w \in L^\infty$, $\|w\| \leq 1$ such that

$$A = \frac{1}{2\pi i} \oint_E w(z) F(z) dz \tag{2.3}$$

Now we put here $F = R$. We get

$$\begin{aligned} 1 =& \frac{1}{2\pi i} \oint_E w(z) R(z) dz \\ =& \frac{1}{2\pi} \oint_E w(z) |R(z) dz| \leq \frac{1}{2\pi} \oint_E |w(z)|\, |R(z) dz| \\ \leq& \frac{1}{2\pi} \oint_E |R(z) dz| \leq 1 \end{aligned}$$

Thus $w = 1$, and hence, by (2.3),

$$A = \frac{1}{2\pi i} \oint_E F(z) dz$$

for all $F \in E^1$, $F(\infty) = 0$.

$\square$

In the end of this section we show that the infimum (2.1) is assumed.

Except for (1.6) one can characterize a Widom domain by the following property. Let $\omega(dx,z)$ be the harmonic measure in the domain with respect to $z\in\Omega$. Then the harmonic measure $\omega(dx):=\omega(dx,\infty)$ is absolutely continuous $\omega(dx)=\rho(x)\,dx$, moreover [16]

$$\int_E \rho(x)\log\rho(x)\,dx>-\infty.$$

So we can define an outer function $\Phi(z)$ multi–valued in Ω, such that

$$\frac{1}{2\pi}|\Phi(z)|=e^{\int_E \log\rho(x)\omega(dx,z)},$$

i.e., $\rho(x)=\frac{1}{2\pi}|\Phi(x)|$, $|\Phi(x)|=\lim_{\epsilon\to 0}|\Phi(x\pm\epsilon i)|$ for a.e. $x\in E$. By α_1 we denote the character generated by $\Phi^{-1}(z(\zeta))$,

$$\Phi^{-1}(z(\gamma(\zeta)))=\alpha_1(\gamma)\Phi^{-1}(z(\zeta)).$$

Let us use an alternative description of the space $E^1_0(\Omega)$.

Proposition 2.3. *Let*

$$H^1_0(\alpha_1):=\{f\in H^1: f(\gamma(\zeta))=\alpha_1(\gamma)f(\zeta),\ f(0)=0\}.$$

$f\in H^1_0(\alpha_1)$ if and only if $f(\zeta)=F(z(\zeta))$ and $F\Phi\in E^1_0(\Omega)$.

Proof. It follows from the fact that one can identify $f\in H^1(\alpha_1)$ with a multi–valued Smirnov class function $F(z)$, $F(z(\zeta))=f(\zeta)$, which is integrable with respect to the harmonic measure

$$\int_{\mathbb{T}}|f(t)|dm(t)=\oint_E|F(x)|\omega(dx)=\frac{1}{2\pi}\oint_E|F(x)\Phi(x)||dx|.$$

Note also that $F\Phi$ is singlevalued in Ω by the definition of the character α_1. ☐

Lemma 2.4. *There exists $H\in E^1_0(\Omega)$ such that*

$$M=\|H\|.$$

Proof. In an extremal sequence we chose a subsequence that converges uniformly on compact subsets in Ω, that is,

$$H(z)=\lim F_n(z),\quad z\in\Omega.$$

In other words $f_n(\zeta) = F_n\Phi^{-1}(z(\zeta))$ converges to $h(\zeta) = H\Phi^{-1}(z(\zeta))$ uniformly on compact subsets in $\mathbb{D}$. Therefore

$$\int_{\mathbb{T}} |h(rt)|dm(t) = \lim_{n\to\infty} \int_{\mathbb{T}} |f_n(rt)|dm(t) \le \lim_{n\to\infty} \|f_n\|.$$

Thus $h \in H_0^1(\alpha_1)$, moreover $\|h\| \le \lim_{n\to\infty} \|f_n\|$. In other words $H \in E_0^1(\Omega)$ and $\|H\| \le \lim_{n\to\infty} \|F_n\|$.

Therefore we get

$$\inf \|F(x)\| \le \|H(x)\| \le \lim_{n\to\infty} \|F_n\| = \inf \|F(x)\|.$$

□

3 Reduction to a weighted L^2 extremal function. Structural Theorem

Lemma 3.1. *Among the extremal functions of Problem* 2.1 *there is a function of the form*

$$H(z) = -\frac{1}{\sqrt{(z-a_0)(z-b_0)}} \prod_{j\ge 1} \sqrt{\left(\frac{z-a_j}{z-b_j}\right)^{\tilde{\delta}_j}} K^2(z), \;\; \tilde{\delta}_j = \pm 1, \tag{3.1}$$

where K is a single–valued Smirnov class function, $K(\infty) = 1$.

Proof. We can assume that $H(z) \in \mathbb{R}$, for $z \in \mathbb{R} \setminus E$, otherwise we use the extremal function $\frac{H(z)+\overline{H(\bar z)}}{2}$. Let us prove some properties of such a function H.

$H(z)$ has not more than one zero in each gap (a_j, b_j). Indeed, if $H(z_1) = H(z_2) = 0$, $a_j < z_1 < z_2 < b_j$, then the function

$$H(z)\left(1 - \frac{\epsilon}{(z-z_1)(z-z_2)}\right)$$

has the smaller E^1-norm for a sufficiently small ϵ.

In the same way we can show that the extremal function has no complex zeros and also $H(z) \ne 0$ for $z \in (a_0, \infty) \cup (-\infty, b_0)$.

Thus the extremal function is of the form (see (1.2))

$$H(z) = R(z, \{x_j\})F(z), \;\; F(\infty) = 1, \tag{3.2}$$

where $x_j \in [a_j, b_j]$ and $F(z)$ is of Smirnov class, $F(z) \neq 0$ in $\mathbb{C} \setminus E$ and such that

$$\begin{aligned} &\frac{1}{2\pi} \oint_E |F(x)||R(x, \{x_j\})|\, dx \\ &= \frac{1}{\pi} \int_E |F(x)| \prod_{j \geq 1} \frac{x - x_j}{\sqrt{(x-a_j)(x-b_j)}} \frac{dx}{\sqrt{(a_0 - x)(x - b_0)}} < \infty. \end{aligned} \tag{3.3}$$

Assume now, that $x_k \in (a_k, b_k)$ for a certain k, i.e., it is not one of the end points. For all other parameters frizzed (including the function F) we consider the integral in (3.3) as a function of the given x_k. It is well defined in this domain ($x_k \in (a_k, b_k)$), moreover it represents a linear function of x_k. Therefore the infimum is assumed on the left or right boundary point of the interval. That is, the extremal function is of the form

$$H(z) = -\frac{1}{\sqrt{(z-a_0)(z-b_0)}} \prod_{j \geq 1} \sqrt{\left(\frac{z-a_j}{z-b_j}\right)^{\delta_j}} F(z), \ \ \delta_j = \pm 1. \tag{3.4}$$

The function F has no zeros and it is single–valued in the domain. Let γ_j be a contour that starts at the "upper" bound of the interval (a_j, b_j), and goes through infinity to the "lower" bound. Then the change of its argument along the contour is of the form

$$\Delta_{\gamma_j} \arg F = 2\pi n_j.$$

Now we represent F in the form

$$F(z) = \prod_{\{j: n_j \text{ is odd}\}} \left(\frac{z-a_j}{z-b_j}\right)^{-\delta_j} \tilde{F}(z)$$

The point is that $\sqrt{\tilde{F}(z)}$ is single–valued in Ω and we set $K(z) = \sqrt{\tilde{F}(z)}$. Thus the lemma is proved. □

The following lemma is almost evident. For a given weight w we define $E^2_w = E^2_w(\Omega)$ as a set of single–valued Smirnov class functions, which are square–integrable against $w|dx|$, i.e.:

$$\|F\|^2_w = \frac{1}{2\pi} \oint_E |F|^2 w |dx| < \infty. \tag{3.5}$$

Lemma 3.2. *Let $\chi(x)$ be the weight function of the form*

$$\chi(x) = \frac{1}{\sqrt{(a_0 - x)(x - b_0)}} \prod_{j \geq 1} \sqrt{\left(\frac{x - a_j}{x - b_j}\right)^{\tilde{\delta}_j}}, \tag{3.6}$$

which corresponds to the particular choice of x_j given by (3.1)*. Let k be the reproducing kernel in this space with respect to ∞. Then*

$$K(z) = \frac{k(z)}{k(\infty)}. \tag{3.7}$$

Proof. For every function $F \in E^2_\chi$, $F(\infty) = 0$, we have

$$\min_\epsilon \oint_E |K(x) + \epsilon F(x)|^2 \chi(x)\, dx = \oint_E |K(x)|^2 \chi(x)\, dx.$$

Therefore

$$\oint_E \overline{K(x)} F(x) \chi(x)\, dx = 0. \tag{3.8}$$

We write

$$K(z) = \frac{k(z)}{k(\infty)} + \tilde{K}(z),$$

where, evidently, $\tilde{K}(\infty) = 0$. Putting $F = \tilde{K}$ in (3.8) we get $\|\tilde{K}\|_\chi = 0$. $\square$

Now we use a description (1.7) of reproducing kernels in H^2–spaces in Widom domains, for details see [19].

Theorem 3.3. *Let $B(z, x_j)$ be the complex Green function, that is, a multi–valued analytic function in Ω such that $-\log|B(z, x_j)| = G(z, x_j)$. Then there exists an extremal functions of the Problem* 2.1*, which is of the form*

$$H(z) = -\frac{1}{\sqrt{(z - a_0)(z - b_0)}} \prod_{j \geq 1} \frac{z - x_j}{\sqrt{(z - a_j)(z - b_j)}} \frac{I(\infty)}{I(z)}, \tag{3.9}$$

where $I(z) = \prod_j B_{x_j}(z)$ is single–valued in Ω.

Proof. Similar to Proposition 2.3, E^2_χ corresponds to the H^2-space with a suitable character, that is, we can relate the scalar product and the reproducing kernel K in E^2_χ with the scalar product and the reproducing kernel K^D with respect to the harmonic measure:

$$\int_E |K(x)|^2 \chi(x) dx = C \int_E |K^D(x)|^2 \prod_{j \geq 1} \frac{x - c_j}{\sqrt{(x - a_j)(x - b_j)}} \frac{dx}{\sqrt{(x - a_0)(b_0 - x)}}$$

with a certain $D \in D(E)$ and $C > 0$. It implies

$$K(z) = \prod_{j \ge 1} \sqrt{\frac{z - x_j}{\sqrt{(z-a_j)(z-b_j)}} \left(\frac{z-b_j}{z-a_j}\right)^{\tilde{\delta}_j/2} \frac{B_{x_j}(\infty)}{B_{x_j}(z)}} \prod_{j \ge 1} \left(\frac{B_{x_j}(z)}{B_{x_j}(\infty)}\right)^{\frac{1+\epsilon_j}{2}}. \tag{3.10}$$

Recall that $K(\infty) = 1$. Since $K(z)$ has no zeros in Ω we have $\epsilon_j = -1$. We substitute (3.10) in (3.1) to get (3.9). Since $K(z)$ is singlevalued in Ω the product $I(z)$ is also singlevalued.

□

4 Singular components of the reflectionless measures and DCT

Proposition 4.1. *Assume that one of the R functions* (1.2) *contains a singular component in its integral representation* (1.1). *Then DCT fails.*

Proof. Since

$$R(z) = \int \frac{d\sigma_s(x)}{x - z} + \frac{1}{\pi} \int_E \frac{|R(x)|dx}{x - z}$$

we have

$$\frac{1}{\pi} \int_E |R(x)|dx = 1 - \sigma_s(E).$$

Thus M in (2.1) is less than 1 if $\sigma_s(E) > 0$. □

The simplest example of a Widom domain where DCT fails was constructed in this way [11], see also [13]: assume that E is a system of intervals, which accumulate to b_0 only. In this case there exists a reflectionless measure with a nontrivial masspoint at b_0 if and only if

$$\int_E \frac{dx}{x - b_0} < \infty. \tag{4.1}$$

So the main point of the example was to demonstrate that (4.1) does not contradict the Widom condition.

Note that a much more advanced example of a reflectionless measure with the *singular continuous component* was constructed in [13]. Conditions that ensure that all reflectionless measures have no singular component were studied in [15].

Now we show that already in this simplest case (4.1) we have

$$M < \inf_{R(z,\{x_j\})} \int_E |R(x)|dx. \tag{4.2}$$

Moreover, in the next section we demonstrate that Proposition 4.1 cannot be inverted: the absence of a singular component for all reflectionless measures does not guarantee DCT for a Widom domain.

To prove (4.2) we define

$$\Pi(z) = -\sqrt{\prod_{j\geq 0} \frac{z-b_j}{z-a_j}} = (z-b_0)R(z,\{b_j\}). \tag{4.3}$$

Note that (4.1) implies that the following limit exists

$$\lambda_* := -\lim_{x\uparrow b_0} \Pi(x) = e^{-\frac{1}{2}\int_E \frac{dx}{x-b_0}} > 0$$

and represents the biggest possible value of the mass for reflectionless measures, i.e.:

$$\inf_R \frac{1}{\pi}\int_E |R(x)|dx = 1-\lambda_*. \tag{4.4}$$

For $\lambda \in (0,\lambda_*)$ we define

$$R_\lambda(z) = \frac{1}{1-\lambda^2}\frac{\Pi^2(z)-\lambda^2}{(z-b_0)\Pi(z)}, \quad I_\lambda(z) = \frac{\Pi(z)+\lambda}{\Pi(z)-\lambda}. \tag{4.5}$$

Here $R_\lambda(z)$ is of the form (1.2) and $I_\lambda(z)$ is the Blaschke product of the form given in Theorem 3.3. For the first function we have

$$R_\lambda(z) = \frac{\sigma_{0,\lambda}}{b_0-z} + \frac{1}{\pi}\int_E \frac{|R_\lambda(x)|dx}{x-z}.$$

Moreover

$$\sigma_{0,\lambda} = \lim_{x\uparrow b_0}(b_0-x)R_\lambda(x) = \frac{\lambda_*^2-\lambda^2}{\lambda_*(1-\lambda^2)}$$

and therefore

$$\frac{1}{\pi}\int_E |R_\lambda(x)|dx = 1 - \frac{\lambda_*^2-\lambda^2}{\lambda_*(1-\lambda^2)} = \frac{1-\lambda_*}{\lambda_*}\frac{\lambda_*+\lambda^2}{1-\lambda^2}.$$

Thus for

$$H_\lambda(z) := R_\lambda(z)\frac{I_\lambda(\infty)}{I_\lambda(z)}, \quad I_\lambda(\infty) = \frac{1-\lambda}{1+\lambda},$$

we get

$$\frac{1}{\pi}\int_E |H_\lambda(x)|dx = \frac{|I_\lambda(\infty)|}{\pi}\int_E |R_\lambda(x)|dx = \frac{1-\lambda_*}{\lambda_*}\frac{\lambda_*+\lambda^2}{(1+\lambda)^2}.$$

The last function decreases with λ, so the smallest value corresponds to $\lambda = \lambda_*$. On the other hand the extremum (4.4) corresponds to $\lambda = 0$. Thus

$$M_1 = \frac{1}{\pi}\int_E |H_{\lambda_*}(x)|dx = \frac{1-\lambda_*}{1+\lambda_*} \tag{4.6}$$

and

$$\inf_R \frac{1}{\pi}\int_E |R(x)|dx = \frac{1}{\pi}\int_E |H_0(x)|dx = 1-\lambda_* > M_1. \tag{4.7}$$

Remark 4.2. Let

$$\int_E \frac{1}{(x-b_0)^n}dx < \infty. \tag{4.8}$$

In this case the extremum is less than (4.6). This smaller value M_n can be expressed by means of a suitable finite (depending on n) moment problem.

5 No DCT: an infinite dimensional defect space related to a single singular point

We start with the trace of the following matrix function

$$\mathfrak{A}(z) = \begin{bmatrix} \cos t\sqrt{z} & \frac{\sin t\sqrt{z}}{\sqrt{z}} \\ -\sqrt{z}\sin t\sqrt{z} & \cos t\sqrt{z} \end{bmatrix} \begin{bmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{bmatrix}, \ t > 0. \tag{5.1}$$

It is easy to check that $\det \mathfrak{A}(z) = 1$ and

$$\frac{\mathfrak{A}^*(z)J\mathfrak{A}(z) - J}{z-\bar{z}} \geq 0, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{5.2}$$

For this reason $\Delta(z) = \frac{1}{2}\mathrm{tr}\mathfrak{A}(z)$ is an entire function with the real ± 1 points. Moreover the function

$$\lambda(z) = \frac{\Delta(z) + \sqrt{\Delta(z)^2 - 1}}{2}$$

is well defined in the upper half-plane, see e.g. [20]. This is the eigenvalue of $\mathfrak{A}(z)$ with the characteristic property $|\lambda(z)| > 1$. So, if we define the domain $\mathbb{C} \setminus E_1$, $E_1 = \{x \in R : |\Delta(x)| \leq 1\}$, then $\log \lambda(z)$ represents the complex Martin function of this domain with the singular point at infinity. In [12, VIII] $\log |\lambda(z)|$ is called the Phragmén–Lindelöf function of the domain.

To simplify further consideration we define $E = E_1 \cup [0, \infty)$.

Lemma 5.1. *For the given E the domain $\mathbb{C}\setminus E$ is of Widom type.*

Proof. First of all we use Theorem [12, p. 407]. Since $\log|\lambda(iy)| = |y| + o(|y|)$, $y \to \pm\infty$, according to this theorem

$$\int_{-\infty}^{\infty} G_1(x,i)dx < \infty,$$

where $G_1(z,i)$ is the Green function of the domain $\mathbb{C}\setminus E_1$. Thus for the extended boundary E we have also

$$\int_{-\infty}^{\infty} G(x,i)dx < \infty, \tag{5.3}$$

where $G(z,i)$ is the Green function of the domain $\mathbb{C}\setminus E$.

Using the explicit formula for

$$\Delta(z) = \cos t\sqrt{z}\cos z - \frac{z+1}{2}\frac{\sin t\sqrt{z}}{\sqrt{z}}\sin z \tag{5.4}$$

we conclude that on the negative half-axis E consists of a system of intervals $[b_{k+1}, a_k]$ close to the points $-k\pi$ (the leading term in asymptotics) of the length $a_k - b_{k+1} \sim \frac{e^{-t\sqrt{k\pi}}}{\sqrt{k\pi}}$.

Comparing in the standard way (5.3) with the common area of triangles built on the intervals (a_k, b_k) with the vertex $(c_k, G(c_k, i))$, where c_k is the critical point of $G(z,i) + G(z,-i)$, we get

$$\sum G(c_k, i)\frac{b_k - a_k}{2} < \infty.$$

Since $b_k - a_k \geq \delta > 0$ (in fact, $b_k - a_k \to \pi$) we obtain the Widom condition $\sum G(c_k, i) < \infty$. $\square$

Proposition 5.2. *For $\Delta(x)$ given by* (5.4)*, let the system of intervals (a_k, b_k) be defined by the condition $|\Delta(x)| > 1$ on the negative half–axis. Then the DCT does not hold in the Widom domain $\Omega = (\mathbb{C}\setminus\mathbb{R})\cup_k (a_k, b_k)$.*

Proof. Let us chose a normalization point in a gap, say, the critical point $c_1 \in (a_1, b_1)$. We claim that the function $F(z) = \cos\tau\sqrt{c_1} - \cos\tau\sqrt{z}$, $0 < \tau \leq t$, belongs to the class E_0^1 (with respect to c_1), that is, it is of Smirnov class in Ω and

$$\int_E \frac{|F(x)|}{|x - c_1|^2}dx < \infty. \tag{5.5}$$

If so, by the Cauchy Theorem,

$$\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{F(z)}{(z-c_1)^2}dz=\tau\frac{\sin\tau\sqrt{c_1}}{2\sqrt{c_1}}\neq 0,$$

but, in fact, this integral vanishes just due to the symmetry $F(x+i0)=F(x-i0)$.

We note that $e^{i\tau\sqrt{z}}$ is in absolute value less than one in Ω and does not vanish. Since the Phragmén–Lindelöf function of the domain behaves as $\operatorname{Im} z$ at infinity this function also does not have the singular inner factor (with the only possible singular point at infinity). Thus it is an outer function in the domain. Therefore $\cos\tau\sqrt{z}$ is of Smirnov class in Ω.

The size of the intervals $[b_{k+1},a_k]$ guaranties that the integral (5.5) converges. The proposition is proved. $\square$

Remark 5.3. Note that in the current example $\int_E\frac{dx}{1+|x|}=\infty$, compere (4.1). Since infinity is the only possible support for a mass-point, there is no reflectionless measure with a singular component, see Sect. 4. Evidently, this set is weakly homogeneous, so we can also refer to the general result [15]. Thus $\inf_R\frac{1}{\pi}\int_E|R(x)|dx=1$, but DCT fails and $M=M(E)<1$ in Problem 2.1.

Remark 5.4. Let us mention that the example in [11], which was discussed in Sect. 4, corresponds to the case when the defect space $H^2(\alpha)\ominus\check{H}^2(\alpha)$ (for a certain α) has dimension one (in particular, it is non–trivial). The example with a singular measure [13] corresponds to an infinite defect space, but with infinitely many "singular" points in the domain. Our example corresponds to *an infinite-dimensional defect space related to a single singular point* in the domain. This remark explains the main idea of the current constuction. The form of the product (5.1) is dictated by a simple reason: each defect space generates a factor with a smaller grow nearby the singular point comparably with the grow of the Martin function of the corresponding domain (in the given case $O(\sqrt{z})$ and $O(z)$ respectively). We will discuss such relations in details in a forthcoming paper.

6 Widom domain with DCT and non-homogeneous boundary

First of all we note that Theorem 3.3 was stated for a bounded set E. We will consider a domain with a unique accumulation point, say b_0, for the

ends of intervals a_k's and b_k's. It is convenient to send this point to infinity by the change of the variable $z_1 = \frac{1}{b_0 - z}$. In this case a function with the only possible singularity at b_0 becomes a standard entire function.

Let E be an unbounded closed set. Note that an unbounded closed set E is homogeneous if (1.10) holds for all $x \in E$ and *for all* $\delta > 0$. In [19] it was shown that homogeneity of the boundary of the Denjoy domain implies DCT. In this section we show that homogeneity is not a necessary condition for DCT.

Lemma 6.1. *Let E be a system of intervals on the negative half-axis which accumulate to infinity only. Assume that $\Omega = \mathbb{C} \setminus E$ is of Widom type and DCT fails. Then there exists an entire function F of Smirnov class in Ω such that*

$$\int_E |F(x)| \frac{dx}{|x|} < \infty. \tag{6.1}$$

Proof. If $\int_E \frac{dx}{|x|} < \infty$ then $F(x) = 1$. If not, then there is no reflectionless measure with a singular component. Thus, the factor $I(z)$ in Theorem 3.3 is not a constant. We define

$$F(z) = \frac{I(0)}{\sqrt{1 - z/a_0}} \prod_{j \geq 1} \frac{1 - z/x_j}{\sqrt{(1 - z/a_j)(1 - z/b_j)}} \left\{ \frac{1}{I(z)} - I(z) \right\}. \tag{6.2}$$

It is of Smirnov class in Ω, moreover it is real valued on the whole real axis. Since $\frac{1}{I(x \pm i0)} - I(x \pm i0) = 2i \mathrm{Im} \frac{1}{I(x \pm i0)}$, the integral (6.1) converges and all possible singularities on E are removable. Thus $F(z)$ is an entire function. □

Remark 6.2. Recall that $|I(z)| < 1$ in the domain. Thus $F(z)$ has only real zeros, moreover only in the set E. Note also that as soon as $F(z)$ is not a constant there exists $F_1(z)$ such that $\int_E |F_1(x)| dx < \infty$. For instance $F_1(z) = \frac{F(z)}{z - x_0}$, where x_0 is a zero of $F(z)$.

We denote by $\mathcal{M}_E(z)$ the Martin function for $\mathbb{C} \setminus E$ with singularity at infinity. The function $\mathcal{M}_E(z)$ has at most order $1/2$, that is, $\mathcal{M}_E(z) = O(\sqrt{|z|})$. We say that a set E is an Akhiezer-Levin set [2, 5] if

$$\limsup_{|z| \to \infty} \frac{\mathcal{M}_E(z)}{\sqrt{|z|}} > 0.$$

It is worth mentioning that in this case $\lim_{x \to +\infty} \frac{\mathcal{M}_E(x)}{\sqrt{x}}$ exists.

Using the change of variable $z = -z_1^2$ we can work with even functions in the upper half-plane, related, correspondingly, to symmetric subsets of the real axis. In particular, in this case we can refer directly to [12, Sect. VIII].

Theorem 6.3. *Let $E = \mathbb{R} \setminus \cup_{k=-\infty}^{\infty}(a_k, b_k)$, $a_{-k} = -b_k$, consist of uniformly separated intervals, say,*

$$b_k - a_k \geq 1, \quad \text{for all } k. \tag{6.3}$$

Assume that the following "weighted" Widom condition holds true

$$\sum_{c_k \in (a_k, b_k): G'(c_k, i)=0} G(c_k, i)(b_k - a_k) < \infty. \tag{6.4}$$

For an arbitrary $\delta \in (0, 1/2)$ define

$$E_\delta = \mathbb{R} \setminus \cup_{k=-\infty}^{\infty}(a_k + \delta, b_k - \delta). \tag{6.5}$$

If

$$\int_{E_\delta} \frac{dx}{1 + |x|} = \infty, \tag{6.6}$$

then the domain $\Omega_\delta = \mathbb{C} \setminus E_\delta$ is of Widom type with DCT.

Proof. First of all (6.3) and (6.4) imply that $\mathbb{C} \setminus E$ is of Widom class. Further, (6.4) implies $\int_{\mathbb{R}} G(t, z)dt < \infty$. By the Koosis' criterion [12] E is a symmetric Akhiezer-Levin set, that is,

$$\lim_{y \to +\infty} \frac{\mathcal{M}_E(iy)}{y} > 0.$$

Evidently the extended set E_δ (6.5) belongs to the Akhiezer-Levin class and Ω_δ is of Widom class.

Assume that DCT fails. By Lemma 6.1 there exists a non trivial even entire function $\tilde{F}(z)$ of Smirnov class in Ω_δ such that

$$\int_{E_\delta} |\tilde{F}(x)| \frac{dx}{1 + |x|} < \infty.$$

Due to (6.6) $\tilde{F}(z)$ is not a constant and we can find a *non trivial* entire function $F(z)$, see Remark 6.2, such that

$$\int_{E_\delta} |F(x)| dx \leq 1. \tag{6.7}$$

Since $F(z)$ is of Smirnov class in Ω_δ we have

$$\lim_{|z|\to+\infty} \frac{\log|F(z)|}{\mathcal{M}_{E_\delta}(z)} = 0, \quad z = x+iy,\ y \geq c|x|,\ c>0.$$

Moreover, since $F(z)$, in particular, is in the Cartwright class and all its zeros are real, one can show, see e.g. [1, p. 58], that for all $\epsilon > 0$ the following a priori estimate holds true $|F(z)| \leq A(\epsilon)e^{\epsilon|z|}$ in the whole complex plane.

Now for $\delta' < \delta$ we define the entire function

$$H(z) = H(z,\delta') = \int_{-\delta'}^{\delta'} F(z+t)dt.$$

By (6.7) $H(z)$ is uniformly bounded on E, $|H(x)| \leq 1$, $x \in E$, and also possesses the a priori estimate $|H(z)| \leq A(\epsilon)e^{\epsilon|z|}$. We apply the Phragmén–Lindelöf principle to $H(z)$ in the domain $\Omega = \mathbb{C}\setminus E$, see [12, p. 406], to get that in fact $|H(z)| \leq e^{\epsilon \mathcal{M}_E(z)}$. Since ϵ is arbitrary small, $H(z) = H(z,\delta')$ is bounded in the whole complex plane, therefore it is a constant. Since this holds for an arbitrary positive $\delta' < \delta$, $F(z)$ is a constant, and due to (6.7) and (6.6) $F(z) = 0$. The contradiction with the claim that $F(z)$ is a non-trivial function shows that DCT holds in Ω_δ.

□

Example 6.4 (Benedicks' set)**.** Let $p > 1$ and put

$$E_\delta = \cup_{n=1}^{\infty}\{[-n^p-\delta, -n^p+\delta] \cup [n^p-\delta, n^p+\delta]\},$$

$\delta > 0$ being taken small enough so that the intervals figuring in the union do not intersect. This set is not homogeneous and it is of Akhiezer-Levin class due to the Koosis' criterion. In fact, due to Benedicks [12, p. 439]

$$G(x,i) \leq C(\delta)\frac{\log|x|}{|x|^{(p+1)/p}},$$

i.e., it is integrable on the real axis and (6.4) is satisfied.

The set has a finite logarithmic length (6.6). So we modify it by adding each second interval formed by the geometric progression:

$$E_{\delta,q} = E_\delta \cup_{n=1}^{\infty} \{[-q^{2n}, -q^{2n-1}] \cup [q^{2n-1}, q^{2n}]\},\ q>1.$$

The resulting set is still not homogeneous, but satisfies all requirements of Theorem 6.3 (we can add to the set all possible gaps of the standard length less than one, in case such open intervals are present in $\mathbb{R}\setminus E_{\delta,q}$). Thus the DCT holds in $\Omega := \mathbb{C}\setminus E_{\delta,q}$.

Corollary 6.5. *For $p > 1$ let*

$$\tilde{E} = \{0\} \cup \{x : 1/x \in E_{\delta,q}\}, \;\; \delta \in (0,1), \;\; q > 1.$$

Then every reflectionless Jacobi matrix $J \in J(\tilde{E})$ is almost periodic.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, New York, 1965.

[2] N. I. Akhiezer, B. Ja. Levin, *Generalization of S. N. Bernstein's inequality for derivatives of entire functions*, Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1960, 111–165. French translation in Fonctions d'une variable complexe. Problèmes contemporains (ed. A. Marcouchevitch), Gauthier–Villars, Paris, 1962, 109–161.

[3] D. Alpay, M. Mboup, *A characterization of Schur multipliers between character-automorphic Hardy spaces*. Integral Equations Operator Theory 62 (2008), no. 4, 455–463.

[4] A. Borichev, M. Sodin, *The Hamburger moment problem and weighted polynomial approximation on a discrete subsets of the real line*. J. Anal. Math. 76 (1998), 219–264.

[5] A. Borichev, M. Sodin, *Krein's entire functions and the Bernstein approximation problem*. Illinois J. Math. 45 (2001), no. 1,167–185.

[6] A. Borichev, M. Sodin, *Weighted exponential approximation and non-classical orthogonal spectral measures*, arXiv:1004.1795v1, 2010.

[7] J. Breuer, E. Ryckman, M. Zinchenko, *Right limits and reflectionless measures for CMV matrices*. Comm. Math. Phys. 292 (2009), no. 1, 1–28.

[8] L. Carleson, *On H^∞ in multiply connected domains*, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Vol. II, W. Beckner, A. P. Calderón, R. Fefferman, and P. W. Jones (eds.), Wadsworth, CA, 1983, pp. 349–372.

[9] J. Christiansen, B. Simon and M. Zinchenko, *Finite gap Jacobi matrices: An announcement*, J. Comp. Appl. Math. 233 (2009) 652–662.

[10] F. Gesztesy, M. Zinchenko, *Local spectral properties of reflectionless Jacobi, CMV, and Schrödinger operators*. J. Differential Equations 246 (2009), no. 1, 78–107.

[11] M. Hasumi, *Hardy Classes of Infinitely Connected Riemann Surfaces*, Lecture Notes in Math. **1027**, Springer, Berlin, 1983.

[12] P. Koosis, *The Logarithmic Integral. I.* Cambridge etc., Cambridge University Press 1988. XVI+606 pp.

[13] F. Nazarov, A. Volberg, P. Yuditskii, *Reflectionless measures with a point mass and singular continuous component*, Pre-print(2007-11-06) oai:arXiv.org:0711.0948

[14] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Anal. Math. **89** (2003), 113–154.

[15] A. Poltoratski and Ch. Remling, *Reflectionless Herglotz functions and Jacobi matrices*, Comm. Math. Phys. 288 (2009), no. 3, 1007–1021.

[16] Ch. Pommerenke, *On the Green's function of Fuchsian groups*. Ann. Acad. Sci. Fenn. Ser. A I Math. 2 (1976), 409–427

[17] Ch. Remling, *The absolutely continuous spectrum of Jacobi matrices*, arXiv:0706.1101.

[18] B. Simon, *Two extensions of Lubinsky's universality theorem*. J. Anal. Math. 105 (2008), 345–362.

[19] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character–automorphic functions*, J. Geom. Anal. **7** (1997), no. 3, 387–435.

[20] P. Yuditskii, *A special case of de Branges' theorem on the inverse monodromy problem*, Integr. equ. oper. theory 39 (2001), 229–252.

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